**Tensors and stuff**

(note we’re using Einstein summation convention, implicitly; maybe see Appendix at bottom)

**3. Changing Coordinates**

Now let’s consider how the metric and the basis vectors change when we go to a new coordinate system. There are many ways to describe a given geometric surface, i.e. many ways to parameterize it, i.e., many different coordinate systems we could use the describe it. Changing the coordinate system doesn’t change the geometry, i.e. we cannot go from a disk geometry to a hemisphere geometry by changing coordinates. All changing coordinates does is change the coordinate axes describing the geometry. Nonetheless, sometimes certain coordinate systems are preferable to others.

**3a. How the metric (components) changes**

Consider a coordinate transformation:



for instance, going from Cartesian to spherical coordinates:



We would like to know how the metric in the new coordinate system relate to that in the old. We can infer this by looking at the metric and seeing how it transforms under a change of variables. So consider the defining equation for the metric in coordinate system S.



and in coordinate system S.



So the prime on ds´ is there because we are constructing the differential element in the primed coordinate system, independent of the unprimed one. There is a prime on g´αβ for same reasons; it is a set of values that is constructed in the primed coordinate system, independent of how it was in the unprimed coordinate system, and so we should keep g and g´ conceptually distinct. Also, when dealing with two different coordinate systems, it is customary to let greek letters represent the indices of the primed coordinate system, and the latin letters represent the indices of the unprimed coordinate system. Alternately, we could just put primes on the primed indices, but that’s a lot of primes. Anyway, now we’re going to say that we should have ds´ = ds, since the change in distance between two points shouldn’t depend on the coordinate system we use to describe it. Therefore equating these two expressions, and changing variables in the du´α and du´β we get:



and so we have,



Actually want the primed g by itself. But what’s prime and what isn’t is matter of notation. So could say, switching around prime and unprimed,



Or better,



We’ll recognize Xαa as the Jacobian Matrix which, through its determinant (*the* Jacobian) relates a product of differentials in one coordinate system to such a product in the other. It’d be nice to get an equation for how g´αβ relates to gab too. First we’ll write our equation above as a matrix equation (maybe see Appendix). Let Xmn = ∂um/∂u´n. Going to dispense with Greek indices for a second. Then we have:



Taking inverse of both sides,



Now the X guys are:



Can verify inverses are properly given, by multiplying the XX-1 and (XT)(XT)-1 together and verifying we get identity matrix. So then,



Or, in our more conventional notation,



i.e.,



**3b. How basis vectors change**

This is easy to work out starting from above. We have:



which clearly suggests:



So that’s the relationship between new and old covariant basis vectors. Let’s get the relationship between new and old contravariant basis vectors. We’ll apply g´αβ to both sides and of our covariant vector equation we get:



So altogether, we can write:



Finally note for earlier/later convenience that:



since,



So Xαa and Xaα are inverses of each other. Note it also follows from these that we can express the function Xab in its various incarnations as a dot product between basis vectors in different coordinate systems. For instance, start with:



and likewise/in general,



These relationships allow us to consider the change of basis from the perspective of the resolution of identity. For instance (implicit summation over *a* as usual),



This reinforces the idea that all we’re doing is changing bases, but that nothing ‘new’ is being added.

**Example**

Consider two coordinate systems, the usual Cartesian one ua = (x,y), and another related to first via a rotation, uα = (u,v). What are the covariant basis vectors of uα? And verify that they are related to the covariant basis vectors of xi via the relationship above.



So we have:



**Example**

Consider spherical coordinates uα= (r,θ,φ). What are the covariant and contravariant basis vectors? What is the 3D Euclidean metric in spherical coordinates, and what is the line element ds2? Well the coordinate transformation is:



and going the other way:



is the coordinate transformation. So the covariant basis vectors are:



and,



and,



The contravariant basis vectors are:



and,



and finally,



You will observe that these covariant and contravariant basis vectors are proportional to one another. So aside from normalization, they are the same. This is always the case for orthogonal coordinate systems. Now observe that:



I’ll just show this explicitly in two cases. First:



and,



So there. Be careful to note that the basis vectors are not necessarily orthonormal – these certainly weren’t. Finally, the metric is:



as you can verify. And so:



**Example**

Calculate gij for cylindrical coordinates, and (dr)2 consequently as well. We have:





and so:



and it follows that:

